CORK INSTITUTE OF TECHNOLOGY
INSTITIÚID TEICNEOLAÍOCHTA CHORCAÍ

MATHEMATICS

The Lee Fields Medal II: SOLUTIONS

1. Steven is deep in thought looking for positive whole numbers m and n that satisfy the equation
$$20m + 19n = 2020.$$
That is easy says Julie, just take $m = 101$ and $n = 0$. Of course, she's correct. Can you find another solution where $0 < m < 10$?

Solution: There are a number of approaches here. The greedy approach would be just to try $m = 1$, solve for n, and check is it a positive whole number; then try $m = 2$, etc., all the way up to $m = 9$.

We can do a little better if we note that both $20m$ and 2020 are multiples of 20, and thus $19n$ must be also. 19 is a prime and so n must be a multiple of 20. That is $n = 20, 40, 60, 80, 100, \dots$. If n gets too large then $19n$ exceeds 2020. This happens for $n = 120$ where $19n = 2280$. Thus $n = 20, 40, 60, 80$ or 100. We have narrowed the greedy choice from nine options to five. Now write m in terms of n:

$$\begin{aligned} 20m + 19n &= 2020 \\ \underset{-19n}{\Rightarrow} 20m &= 2020 - 19n \\ \underset{\div 20}{\Rightarrow} m = m(n) &= \frac{2020 - 19n}{20}, \end{aligned}$$

Now we examine m for $n = 20, 40, 60, 80, 100$:

$$\begin{aligned} m(20) &= \frac{2020 - 19(20)}{20} = 82 > 10 \\ m(40) &= 63 > 10 \\ m(60) &= 44 > 10 \end{aligned}$$

At this point we see that $m(80) > 10$ but $m(100) = 6 < 10$. The work above shows that this solution is unique: $(m, n) = (6, 100)$.

2. The following holds for any real numbers a, and r such that $|r| < 1$:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

Hence, or otherwise, express

$$0.42424242\cdots$$

as a rational number.

Solution: First of all, the calculator cannot handle the infinite here, that while it will give us

$$q_1 := \frac{14}{33} \text{ “=” } 0.4242424242,$$

it also gives us

$$q_2 := \frac{424242424242427}{999999999999999} \text{ “=” } 0.4242424242.$$

And these two fractions are not the same. However you put $\frac{14}{33}$ into the calculator and get $0.\dot{4}\dot{2} = 0.\overline{42}$... isn't that enough. Well, no, the calculator also thinks $q_2 = 0.\overline{42}$, but it is not equal to that, it is equal to (as a decimal):

$$0.\overline{424242424242427} \neq 0.\overline{42}.$$

Therefore just writing down $\frac{14}{33}$ without justification is not a complete answer. In general, machines cannot handle the infinite (or indeed the very small or very large).

So how do we approach such a problem? There are two well-known approaches. The first takes advantage of the hint:

$$\begin{aligned}
0.42424242\ldots &= 0.42 + 0.0042 + 0.000042 + 0.00000042 + \cdots \\
&= \frac{42}{100} + \frac{42}{10000} + \frac{42}{1000000} + \frac{42}{100000000} \cdots \\
&= \frac{42}{100} + \frac{42}{100} \cdot \frac{1}{100} + \frac{42}{100} \cdot \left(\frac{1}{100}\right)^2 + \frac{42}{100} \cdot \left(\frac{1}{100}\right)^3 + \cdots \\
&\sim a + ar + ar^2 + ar^3 + \cdots,
\end{aligned}$$

with $a = \frac{42}{100}$ and $r = \frac{1}{100}$ (note $|1/100| < 1$). Thus

$$0.42424242\cdots = \frac{a}{1-r} = \frac{\frac{42}{100}}{1 - \frac{1}{100}} \underset{\times \frac{100}{100}}{=} \frac{42}{100 - 1} = \frac{14}{33}.$$

There is another nice approach; $\alpha := 0.42424242\ldots$ has a *self-similarity* that we can take advantage of. Note that:

$$100\alpha = 42.4242\cdots = 42 + \alpha,$$

so that

$$\begin{aligned}
100\alpha &= 42 + \alpha \\
\Rightarrow 99\alpha &= 42 \\
\Rightarrow \alpha &= \frac{42}{99} = \frac{14}{33}.
\end{aligned}$$

3. Consider a row of n tiles, each tile of which is red or blue. Suppose that a blue tile is never followed by a blue tile, so that $RBRBRR$ is allowed, but $RBBRRB$ is not.

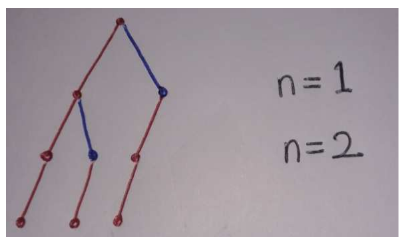
   Let $t(n)$ be the number of allowed tilings. Find $t(1)$, and $t(2)$, and come up with a formula for $t(n)$ in terms of $t(n-1)$ and $t(n-2)$.

   Solution: Let us look at $t(1)$ and $t(2)$ for starters.

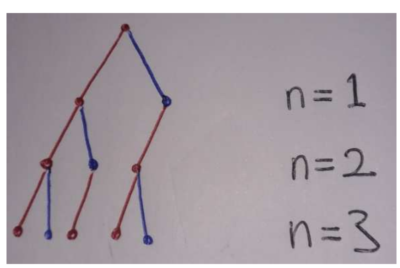
   The 1-tilings are R and B, both allowed, and so $t(1) = 2$.

   Now let us consider $t(2)$. The 2-tilings are RR, RB, BR, and BB. However BB is not allowed and so $t(2) = 3$.

   The next problem needs a good picture to blow it open. Start with what we showed above:



   We start with R and B for $t(1) = 2$. We can put a red onto both of these to get RR and BR. However we cannot draw a blue on both, only on red to give RB. This gives $t(2) = 3$. After this we can put a red on each of the $t(2) = 3$ 2-tilings. This gives RRR, RBR and BRR. We cannot put a blue on every 2-tiling: only the ones that finish red. Call the tilings that 'finish red' by red, and those that 'finish blue' by blue. We can only put a blue on red 2-tilings. They are the 2-tilings that put a red on the 1-tilings, and there are $t(1) = 2$ of those:



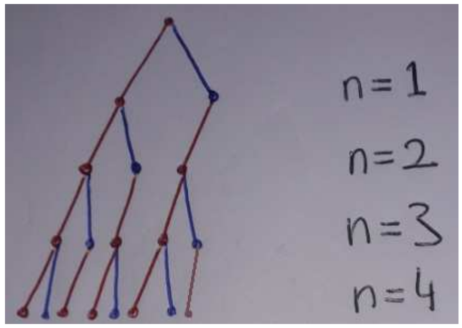
   Thus we see that:

$$t(3) = 3 + 2 = \underbrace{t(2)}_{\text{reds on ALL the previous}} + \underbrace{t(1)}_{\text{blues on all the previous REDs}}.$$

   This suggests an answer of

$$t(n) = t(n-1) + t(n-2).$$

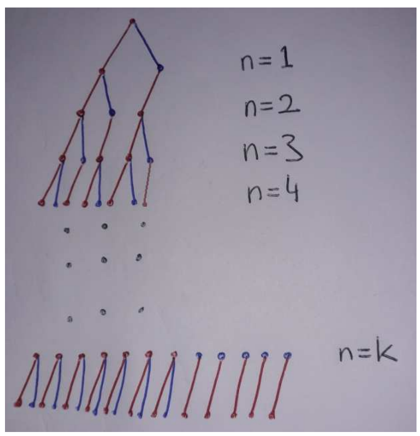
Let us look at $t(4)$:



Here we see the same pattern. Reds on all the $t(3) = 5$ 3-tilings, and blues on the red 3-tilings: and there are $t(2) = 3$ of those. So altogether

$$t(4) = t(3) + t(2) = 5 + 3 = 8.$$

To argue from here to the general isn't very easy. The following should convince you:



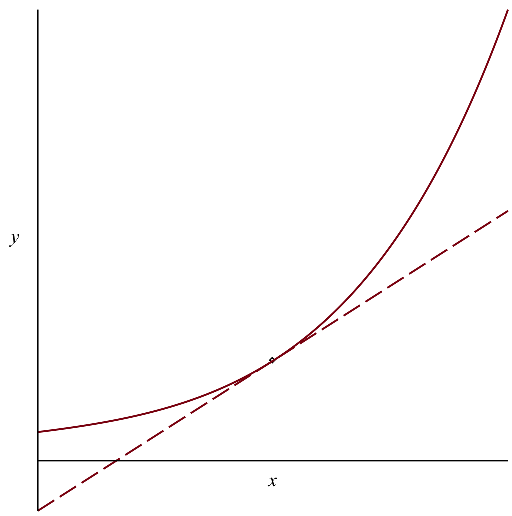
Looking at $n = k$, we have, by definition, $t(k)$ k-tilings, of which $t(k-1)$ are red. We can put reds on all of the k-tilings: this gives $t(k)$ red $(k+1)$-tilings. We can also put blue tiles on all the $t(k-1)$ red k-tilings. Therefore

$$t(k+1) = t(k) + t(k-1),$$

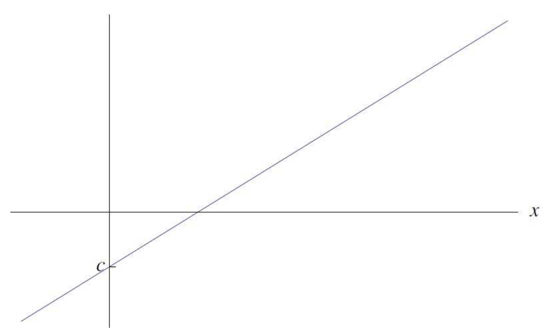
and thus the (*recursive*) relation above holds. The number of tilings are *Fibonacci Numbers*.

4. Suppose that m and c are constants. What is the equation of the tangent to the graph $y = mx + c$ at $x = 1$?

Solution: To answer this question well one must understand what a tangent is: the tangent to a curve at a point P is the *best*[1] *line-approximation to the curve* at that point:



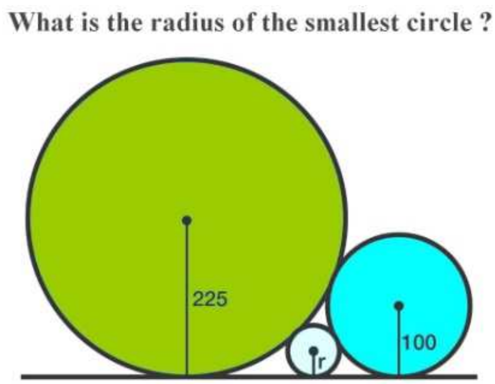
Next we must understand that the graph (of) $y = mx + c$ is a straight-line:



... and the best line approximation to a line — at any point, not just $x = 1$ — is itself. The answer to the question is $y = mx + c$.

[1]this requires some further interpretation

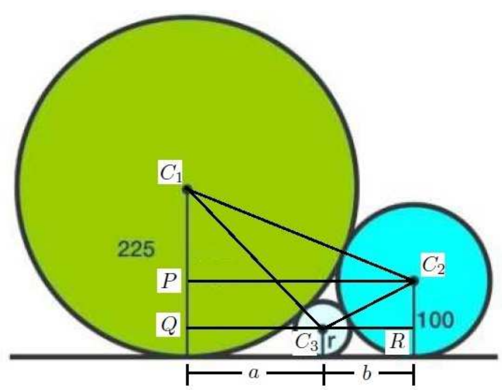
5. Find r



Solution: Three ideas come into this particular solution[2]:

- If circles of radius r_1 and r_2 are touching externally at a single point, then the distance between their centres is $r_1 + r_2$
- In general, we can solve three equations in three unknowns. The radius of the smallest radius r is one. Two others arise in this solution.
- Therefore we need three equations: if we can find three right-angled triangles we can apply Pythagoras Theorem three times.

Consider the following:



Here we have labeled the centres by C_i, and drawn line segments defining points P, Q, R. Finally we have defined two additional variables a and b. We calculate a number of lengths by addition/subtraction, and also by the theorem about touching circles. These yield $|C_1C_2| = 325$, $|C_1P| = 125$, $|C_2P| = a + b$, $|C_1Q| = 225 - r$, $|C_3Q| = a$, $|C_1C_3| = 225 + r$, $|C_2R| = 100 - r$, $|C_3R| = b$, $|C_2C_3| = 100 + r$.

We have three unknowns so perhaps three applications of Pythagoras Theorem to three circles should blow this problem open.

[2]there are probably many more, including, for example, calculating a certain area in more than one way

Consider the right-angled triangles ΔC_1C_2P, ΔC_1C_3Q, and ΔC_2C_3R. This yields:

$$\begin{aligned} 125^2 + (a+b)^2 &= 325^2 \\ (225-r)^2 + b^2 &= (225+r)^2 \\ (100-r)^2 + a^2 &= (100+r)^2 \end{aligned}$$

Multiplying out the second two

$$\begin{aligned} 225^2 - 450r + r^2 + b^2 &= 225^2 + 450r + r^2 \\ 100^2 - 200r + r^2 + a^2 &= 100^2 + 200r + r^2 \end{aligned}$$

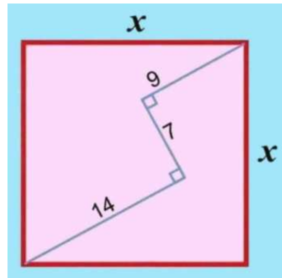
which yield

$$\begin{aligned} 900r = b^2 &\underset{\sqrt{}}{\Rightarrow} b = \pm\sqrt{900r} \underset{a>0}{=} +30\sqrt{r} \\ 400r = a^2 &\underset{\sqrt{}\text{ and } b>0}{\Rightarrow} a = 20\sqrt{r} \end{aligned}$$

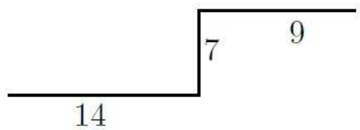
Now substitute these into the first equation:

$$\begin{aligned} 125^2 + (30\sqrt{r} + 20\sqrt{r})^2 &= 325^2 \\ \underset{-125^2}{\Rightarrow} (50\sqrt{r})^2 &= 325^2 - 125^2 \\ \Rightarrow 2500r &= (450)(200) \\ \underset{\div 2500}{\Rightarrow} r &= 36 \end{aligned}$$

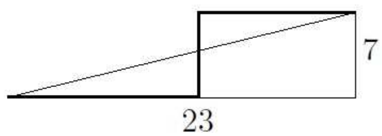
6. Find x



Solution: Look at that 'S' shape and extract it:



Now note that we can form a right-angled triangle out of this 'S':



Now let h be the length of the hypotenuse and apply Pythagoras Theorem:

$$h = \sqrt{23^2 + 7^2} = 17\sqrt{2}$$

Now note that this hypotenuse is the diagonal of the square above, and that the diagonal is the hypotenuse of a right-angled triangle, whose other lengths are both x:

$$\begin{aligned}(17\sqrt{2})^2 &= x^2 + x^2 \\ \Rightarrow 2x^2 &= 578 \\ \underset{\div 2}{\Rightarrow} x^2 &= 289 \\ \underset{\sqrt{}\text{ and } x>0}{\Rightarrow} x &= 17.\end{aligned}$$

7. In a game show you have to choose one of three doors. One conceals a new car and the other two contain angry lions who will attack you. You choose but your chosen door is not opened immediately. Instead the presenter tells you that another door (which you have not picked), contains a lion. You then have the opportunity to change your mind. Is there any benefit to doing so? Justify your answer.

   Solution: This is a famous problem which has sparked argument, debate, and controversy, most famously in the early 1990s when it was posed as a problem in *Parade* magazine. The solution is counterintuitive: not only should you switch, but you *double* your probability of winning if you switch.

   There are many solutions. This is one such solution. "Benefit" here needs some interpretation, but let us make the assumption that there are only two strategies, and the choice is between these two:

   - ALWAYS switch,
   - ALWAYS stay,

   and there is a benefit to switching over staying if the *probability* of winning when you switch is greater than the probability of winning when you stay.

   Suppose you always stay. If you always stay you can only win if you pick the new car at the start. Therefore

$$\mathbb{P}[\text{stay and win}] = \frac{1}{3}.$$

   Now suppose you always switch. The thing is if you pick the car you lose, oh no, but this only happens with probability 1/3. What if you pick a door with a lion behind it? Well, if you pick a door with a lion behind it, the presenter will open the *other* door with a lion behind it. The choice, on this occasion, is between the door you picked originally, with a lion, and a door with the car. If you switch, if you picked a door with a lion, you will win the car. *When you always switch, you always win when you pick a door with a lion...* and this happens with probability 2/3. Therefore:

$$\mathbb{P}[\text{switch and win}] = \frac{2}{3}.$$

   If you repeatedly play this game with the same strategy, by the *Law of Large Numbers*, those who always switch can expect to win the car twice as often as those who stay. Assuming, of course, that the intermittent lion attacks are not fatal!

8. Write down an expression for
$$\frac{d^{2019}}{dx^{2019}}x^{2019},$$
the 2019-th derivative of x^{2019}.

Solution: For brevity let us write $f(x) = x^{2019}$ and $f^{(k)}x$ for the k-th derivative. Let us calculate the first few derivative:

$$\begin{aligned} f^{(1)}(x) &= 2019 \cdot x^{2018} \\ \Rightarrow f^{(2)}(x) &= 2019 \times 2018 \cdot x^{2017} \\ \Rightarrow f^{(3)}(x) &= 2019 \times 2018 \times 2017 \cdot x^{2016} \\ \Rightarrow f^{(4)}(x) &= 2019 \times 2018 \times 2017 \times 2016 \cdot x^{2015} \end{aligned}$$

Note that at this fourth derivative, there are four descending numbers and the power has decreased by four, $2015 = 2019 - 4$. This pattern is not going to change[3]. *Inductively*,

$$f^{(2018)}(x) = \underbrace{2019 \times 2018 \times \cdots \times 3 \times 2}_{\text{2018 factors}} \cdot x^{2019-2018} = 2019 \times 2018 \times \cdots \times 3 \times 2 \cdot x.$$

Differentiate this to get $f^{(2019)}(x)$ aka

$$\frac{d^{2019}}{dx^{2019}}x^{2019} = 2019 \times 2018 \times \cdots \times 2 \times 1 =: 2019!,$$

which is a number too large for the calculator to handle, a 5798 digit number, which starts 19113710....

[3]if you are unconvinced one can use the higher level LC technique of induction on $P(n) : \frac{d^n}{dx^n}x^n = n!$ to prove this

9. If you expand

$$(1+x)^{2019} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2019}x^{2019},$$

what is the coefficient of x^2, a_2?

Solution: Most students used the *Binomial Theorem*, and if they used it properly got the correct answer $2019C2 = 2,037,171$, and full marks, but as that is higher level LC material, there must be solutions that don't use the binomial theorem. Here some relatively similar solutions are given.

First up is by trying to spot a pattern. This is labour intensive:

$$\begin{aligned}
(1+x)^2 &= 1 + 2x + x^2 && \Rightarrow a_2 = 1 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 && \Rightarrow a_2 = 3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 && \Rightarrow a_2 = 6 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 && \Rightarrow a_2 = 10
\end{aligned}$$

Do we spot a pattern? Do we think it will continue:

$$1 \to 3 \to 6 \to 10 \to 15?$$

The question is only really asking you to find this coefficient so if you like you could *guess* that the pattern repeats. Not very assured, but maybe you will get lucky? Let us look at the numbers again, and let N be the power of $(1+x)^N$:

$$\underbrace{1}_{N=2} \to \underbrace{1+2}_{N=3} \to \underbrace{1+2+3}_{N=4} \to \underbrace{1+2+3+4}_{N=5} \to \underbrace{1+2+3+4+5}_{N=6?}?.$$

The guess is that for $N = 2019$,

$$a_2 = 1 + 2 + \cdots + 2018.$$

Evaluating this on the calculator, I estimate, would take about an hour! Maybe as it is presented it is a fine answer... but we are not 100% sure on it so if there was a smart way to calculate this, that would be great. Thankfully a famous mathematician called *Gauss*, when they were young, showed us the way. Write the sum forwards and backwards on top of itself:

$$\begin{array}{ccccccc}
1 & + & 2 & + & \cdots & + & 2018 \\
2018 & + & 2017 & + & \cdots & + & 1
\end{array}$$

Notice that all the columns sum to 2019, and also there are 2018 columns. Therefore if we add up all the column sums we get:

$$2018 \times 2019.$$

However this is twice the sum, and so

$$a_2 = 1 + 2 + \cdots + 2018 = \frac{2018 \times 2019}{2} = 2,037,171.$$

There is still some doubt here... how do we know that the pattern keeps going? We could prove that it does using the higher level proof technique of induction. Instead we will consider a slightly different approach.

Write

$$(1+x)^{2019} = \underbrace{(1+x)\cdot(1+x)\cdot(1+x)\cdots(1+x)}_{2019 \text{ factors}}.$$

To multiply this out fully, you take either 1 or x from each factor, and multiply these all together. For example, if you took the first 1019 xs and 1000 '1's you would have a term x^{1019}. Now make all possible combinations of 1 or x from the first factor, multiplied by 1 or x from the second factor, multiplied..., multiplied by 1 or x from the 2019th factor. $(1+x)^{2019}$ is all of these added together in the same way that, if you carefully multiply out

$$\begin{aligned}(a_1+a_2)\cdot(b_1+b_2)\cdot(c_1+c_2) = a_1b_1c_1 + a_1b_1c_2 + a_1b_2c_1 + a_1b_2c_2 + \\ a_2b_1c_1 + a_2b_1c_2 + a_2b_2c_1 + a_2b_2c_2.\end{aligned}$$

So, how can we get terms of the form x^2? Well, from the 2019 factors we pick two, we pick pairs of factors. These factors supply two xs and from the rest we choose '1's. We can do this as follows, pick factors

$$\begin{aligned}
&\underbrace{(1\&2) \text{ or } (1\&3) \text{ or } \cdots \text{ or } (1\&2019)}_{2018\,x^2}\\
&\underbrace{(2\&3) \text{ or } (2\&4) \text{ or } \cdots \text{ or } (2\&2019)}_{2017\,x^2}\\
&\vdots\\
&\underbrace{(2017\&2018) \text{ or } (2017\&2019)}_{2\,x^2}\\
&\underbrace{(2018\&2019)}_{1\,x^2}
\end{aligned}$$

In this way we have that

$$a_2x^2 = 2018x^2 + 2017x^2 + \cdots + 2x^2 + 1x^2 = (2018 + 2017 + \cdots + 1)x^2 \Rightarrow a_2 = 2,037,171.$$

Actually what is ordinary level LC material is to ask 'how many ways can we make x^2'? Well, 'how many ways from 2019 factors can we choose two' to give an x? 'From 2019 how many ways is there to choose 2'? The answer is $2019C2$ — "2019 choose 2" — your calculator will give $2,037,171$.

Arguing like this leads to a proof of the Binomial Theorem.

This question is really only looking for the answer, and ordinary level Leaving Cert techniques (above) can solve this. However, one student gave a marvelous answer using higher level Leaving Cert techniques. It uses the *Chain Rule* of Differentiation. The student said, OK, suppose that for all x:

$$(1+x)^{2019} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{2019} x^{2019}.$$

Differentiate both sides with respect to x (using the chain rule):

$$2019 \cdot (1+x)^{2018} = a_1 + a_2 \cdot 2x + a_3 \cdot 3x^2 + \cdots + 2019x^{2018}.$$

Differentiate again:

$$2019 \times 2018 \cdot (1+x)^{2017} = 2a_2 + 3a_3 \cdot 2x + \cdots + 2019 \times 2018x^{2017}.$$

If true for all x, substitute $x = 0$:

$$\begin{aligned} 2019 \times 2018 \cdot 1^{2017} &= 2a_2 + 0 + \cdots + 0 \\ \Rightarrow 2a_2 &= 2019 \times 2018 \\ \underset{\div 2}{\Rightarrow} a_2 &= 2,037,171. \end{aligned}$$

VERY clever.

10. Using a 5-litre container and a 7-litre container, what is the minimum supply of water you need to measure exactly 4 litres of water?

Solution: It can be done with 14. Let (a, b) the amount of water in the 7 litre and 5 litre respectively, and $[w]$ the amount of water used. Consider the following:

$$\underbrace{(0,0)}_{\text{initially empty}} \xrightarrow[\text{fill the 7 litre [7]}]{} (7,0) \xrightarrow[\text{switch from the 7 to the 5 litre [7]}]{} (2,5) \xrightarrow[\text{empty the 5 litre [7]}]{} (2,0)$$

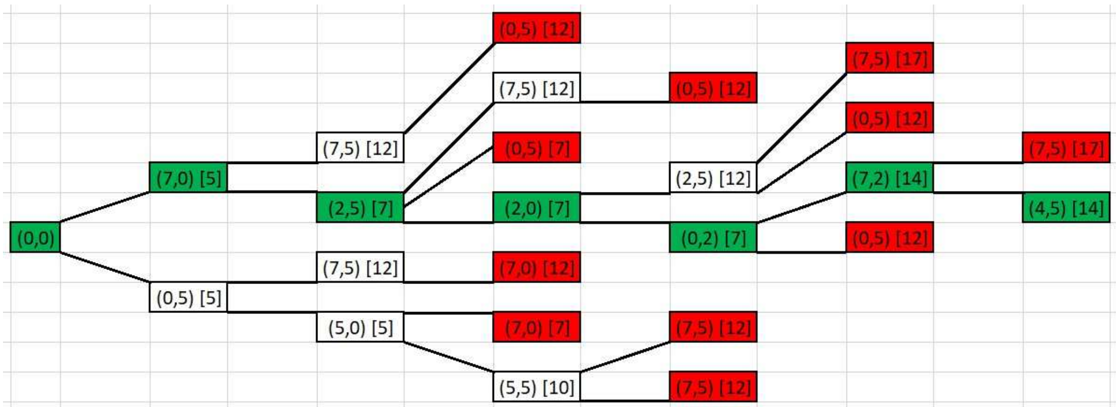
$$\xrightarrow[\text{switch from the 7 to the 5 litre [7]}]{} (0,2) \xrightarrow[\text{fill the 7 litre [14]}]{} (7,2) \xrightarrow[\text{switch from the 7 to the 5 litre [14]}]{} (\mathbf{4}, 5)$$

This is sufficient for full marks.

Can it be done in less? The answer is no. It IS possible to do it (for the first time) using MORE, but those ways involve obvious waste (like filling the 7, emptying it, then filling it again). How do we know this... well, there are only so many things that can be done:

- Fill the 7 litre
- Fill the 5 litre
- Switch water from the 7 litre to the 5 litre
- Switch water from the 5 litre to the 7 litre
- Empty the 7 litre
- Empty the 5 litre

We can list all the 'good' sequences of actions, and the associated 'states', (a, b), and water used, $[w]$. The water used, $[w]$ is non-decreasing as we go from left-to-right, and so we never loop back to a previously visited state. These looped sequences are not 'good'. Also, if we get to a state (a, b) using *more* water than we have done so already, this is a dead end and coloured red. Continuing on does not produce a 'good' sequence. The space of 'good' sequences starts like this:



Note that not only is 14 the answer, but any 'good' sequence of actions must go through this state, so any other answer (e.g. 24 L), necessarily goes through this state.